NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMAL SEARCH PLANS FOR MOVING TARGETS*†

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Let $\{X_t, t > 0\}$ be a stochastic process representing the motion of a target in Euclidean n-space. Search effort is applied at the rate $m(t) > 0$ for $0 < t < T$, and it is assumed that this effort can be distributed as finely as desired over space. We seek an optimal search plan, i.e., an allocation of effort in time and space which maximizes the probability of detecting the target by time T. When the detection function is concave, theorem 1′ of this paper gives necessary and sufficient conditions for an optimal search plan for a class of target motion processes $\{X_t, t > 0\}$ which includes virtually any reasonable model of target motion.

In the special case of a discrete-time target motion process and an exponential detection function, the necessary and sufficient conditions have the following intuitive interpretation: For $t = 0, 1, 2, \ldots, T$, let $\bar{g}_t$ be the probability distribution of the target's location at time t given that the effort at all times before and after t failed to detect the target. The optimal plan allocates the effort for time t so as to maximize the probability of detecting a stationary target with distribution $\bar{g}_t$ within the effort constraint $m(t)$. This special case is a generalization of Brown's (1978) result for discrete time and space target motion.

1. Introduction. In this paper we find necessary and sufficient conditions for a search plan to maximize the probability of detecting a moving target by time T under constraints on the rate at which search effort may be applied. These conditions apply to a very wide variety of moving target problems in continuous or discrete time and continuous or discrete space. Many previous results concerning necessary and sufficient conditions for moving target problems appear as special cases of the results obtained here. In particular our results include the necessary conditions obtained by Hellman (1972) for diffusion processes and by Saretsalo (1973) for continuous time and space Markov processes, and they extend those results by showing the conditions are also sufficient. The results of Stone (1977) and Persiheimo (1976), (1977) for continuous time generalized conditionally deterministic motion are special cases of the results in this paper as well as those of Brown (1978) for target motion in discrete time and space.

However, the results of this paper go far beyond the previous results in this area. When the detection function is concave theorem 1′ of this paper gives necessary and sufficient conditions for an optimal search plan which apply to any target motion process with Borel measurable sample paths and for which the expectation (conditioned on the target passing through a point y at time t) which defines D_T in (3.2) or (3.13) is well defined. Both of these restrictions are very mild, and virtually any reasonable stochastic process which could be used to represent target motion should satisfy them. The major restriction on the results given in this paper is that the searcher must be able to distribute his effort as finely as he desires over space, i.e., effort must be *infinitely divisible in space*. This prevents the direct application of these results to problems such as the search of one submarine for another where the

searcher is required to follow a smooth path and has limited control over the manner in which search effort is distributed about the path.

In contrast to the generality of theorem 1', previous results are limited to exponential detection functions and special types of Markov processes (Hellman (1972) and Saretsalo (1973)) or to very special types of target motion like conditionally deterministic motion (Stone (1977) and Persiheimo (1977)), or to discrete space and time target motion coupled with an exponential detection function.

In the case of a discrete time target motion process and an exponential detection function, the necessary and sufficient conditions have the following simple intuitive interpretation which provides insight into the nature of the optimal allocation and which has provided the basis for efficient numerical calculation of optimal plans for discrete time and space Markov processes (see Brown (1978)). Suppose ψ^* is a plan that maximizes the probability of detection by time T. For $t = 0, 1, \ldots, T$, let $\tilde{g}_t$ be the probability distribution of the target's location at time t given that the search effort at all times other than t failed to detect the target. Then $\psi^*(\cdot, t)$, the allocation of effort at time t under the optimal plan ψ^*, is the same as the allocation of effort which maximizes the probability of detecting a stationary target with probability distribution $\tilde{g}_t$. This observation was first made by Brown (1978) for the case of discrete time and space target motion processes. Stone et al. (1978) gives examples of the use of the above characterization to find optimal search plans for arbitrary discrete time target motion processes. Washburn (1978) has generalized the necessary conditions of Brown to a case where effort is not required to be infinitely divisible in space and shown by example that the conditions are not sufficient when the assumption of infinite divisibility is dropped.

In section §2 we state the optimal search problem that we are considering. The statement of the search problem and the theorems are given in terms of a discrete time and continuous space target motion model. Modifications required to apply the results to discrete space or continuous time are noted after the results are stated. A discrete time model is used because it is most amenable to numerical calculation and because discrete time allows us to present proofs that are considerably simpler than those required for continuous time. A continuous space is chosen for the basic presentation because it illustrates that the results are not simply applications of the Kuhn-Tucker theorem. In addition, the discrete-space results are usually transparent once the continuous-space results have been obtained.

In theorm 1 of §3 we prove that conditions (3.4), stated below, are necessary and sufficient for the optimality of search plan in discrete time and continuous space when the detection function is concave. In the case of continuous time we observe that the necessary conditions (3.4) are true but their proof is not a simple extension of the one given in theorem 1. Instead the reader is referred to the proof of theorem 5.2 of Stone (1977) which may easily be applied to proving this result. However, the proof in Stone (1977) is very technical involving a demonstration of the existence of a measurable selection from a function space. This difficulty is not present in discrete time. In theorem 1' of §3 we present a unified statement of the necessary and sufficient conditions which applies to any combination of discrete or continuous space and time.

§4 considers the special case of a discrete time target motion process and an exponential detection function. For this case we give a simple proof of the necessity result which relies only on the necessary conditions for an optimal search plan for a stationary target.

2. Problem statement. Let $\{X_t; t = 0, 1, \ldots, T\}$ be a discrete time stochastic process where X_t takes values in Euclidean n-space, Y, and T is a positive integer. The

random variable X_t represents the target's position at time t. We assume that $(X_0, X_1, \ldots, X_T)$ has a joint density function p defined on Y^{T+1}. Let p_t be the density of the marginal distribution of X_t for $t = 0, 1, \ldots, T$.

Let $\psi(\cdot, t)$ be the allocation of search density at time t for $t = 0, 1, \ldots, T$. That is, $\psi(y, t)$ is the effort density applied to point y at time t under plan ψ. We assume that ψ is a member of the space F of real-valued Borel functions f defined on $Y \times \{0, 1, \ldots, T\}$ such that

$$\int_Y |f(y, t)| \, dy < \infty \quad \text{for } t = 0, 1, \ldots, T,$$

$$\|f\| \equiv \sum_{t=0}^{T} \operatorname*{ess\,sup}_{y \in Y} |f(y, t)| < \infty.$$

Then F is a linear space with norm $\|f\|$ for $f \in F$. Let $F^+ = \{f \in F : f \geqslant 0\}$, and let $m(t) > 0$ be the amount of effort which is available at time t for $t = 0, 1, \ldots, T$. Define

$$\Psi(m) = \left\{ \psi \in F^+ : \int_Y \psi(y, t) \, dy = m(t) \text{ for } t = 0, 1, 2, \ldots, T \right\}. \tag{2.1}$$

Let $X(\omega, \cdot)$ denote a sample path of the process X. If the target follows this path and we allocate search effort according to the plan ψ, then

$$b\left(\sum_{s=0}^{T} \psi(X(\omega, s), s) \right)$$

is the probability of detecting the target by time T. The function b is called a detection function. It relates the accumulated search density along the target's path to probability of detection. Letting E indicate expectation over sample paths, we have that the overall probability of detection by time T is

$$P_T[\psi] \equiv E\left[b\left(\sum_{s=0}^{T} \psi(X(\omega, s), s) \right) \right].$$

In the sequel we shall usually not indicate the dependence of X on ω.

The optimal search problem under consideration is to find $\psi^* \in \Psi(m)$ such that

$$P_T[\psi^*] = \max\{ P_T[\psi] : \psi \in \Psi(m) \}. \tag{2.2}$$

A plan $\psi^* \in \Psi(m)$ that satisfies (2.2) is called *T-optimal within* $\Psi(m)$.

3. Necessary and sufficient conditions. In this section we find necessary and sufficient conditions for a plan ψ^* to be *T*-optimal within $\Psi(m)$ when the detection function b is concave. The conditions involve the Gateaux differential of P_T which we will now define and calculate under the assumption that for some finite $\kappa > 0$, the derivative b' of b satisfies $0 \leqslant b'(z) \leqslant \kappa$ for $z \geqslant 0$.

Gateaux differential of P_T. Let $\psi, h \in F$. If

$$P_T'[\psi, h] \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(P_T[\psi + \epsilon h] - P_T[\psi] \right)$$

exists, then $P_T'[\psi, h]$ is the *Gateaux differential* of P_T at ψ in the direction h.

For $\psi \in F^+$, let $K(\psi)$ be the cone of directions h such that $\psi + \theta h \in F^+$ for all

sufficiently small nonnegative values of θ. Now for $\psi \in F^+$ and $h \in K(\psi)$,

$$P_T'[\psi, h] = \lim_{\epsilon \to 0} E\left\{ \frac{1}{\epsilon}\left[b\left(\sum_{s=0}^{T} \psi(X_s, s) + \epsilon h(X_s, s)\right) - b\left(\sum_{s=0}^{T} \psi(X_s, s)\right)\right]\right\}.$$

Since the integrand is bounded by $\kappa\|h\|$, we may invoke the dominated convergence theorem to obtain

$$P_T'[\psi, h] = E\left\{ b'\left(\sum_{s=0}^{T} \psi(X_s, s)\right) \sum_{t=0}^{T} h(X_t, t)\right\}$$

$$= E\left\{ \sum_{t=0}^{T} b'\left(\sum_{s=0}^{T} \psi(X_s, s)\right) h(X_t, t)\right\}. \tag{3.1}$$

Let p_t denote the marginal density for X_t, and let E_{yt} denote expectation conditioned on $X_t = y$. Then for $\psi \in F^+$ and $h \in K(\psi)$,

$$P_T'[\psi, h] = \sum_{t=0}^{T} \int_Y E_{yt}\left[b'\left(\sum_{s=0}^{T} \psi(X_s, s)\right)\right] h(y, t) p_t(y)\, dy.$$

Note that we have expressed $P_T'[\psi, \cdot]$ as a linear functional on $K(\psi)$. Define

$$D_T(\psi, y, t) = E_{yt}\left[b'\left(\sum_{s=0}^{T} \psi(X_s, s)\right)\right] p_t(y) \quad \text{for } \psi \in F^+, y \in Y, t = 0, 1, \ldots, T. \tag{3.2}$$

Then

$$P_T'[\psi, h] = \sum_{t=0}^{T} \int_Y D_T(\psi, y, t) h(y, t)\, dy \quad \text{for } h \in K(\psi). \tag{3.3}$$

Necessary and sufficient conditions. We now state and prove necessary and sufficient conditions for T-optimality. Let $\mathscr{E}_T$ be Euclidean T-space and $\mathscr{E}_T^+$ be the nonnegative orthant in $\mathscr{E}_T$.

THEOREM 1. *Suppose that b is concave and that it has a bounded nonnegative derivative, b'. Then ψ^* is T-optimal within $\Psi(m)$ if and only if there exists $(\lambda(0), \ldots, \lambda(T)) \in \mathscr{E}_{T+1}^+$ such that*

$$D_t(\psi^*, y, t) = \lambda(t) \quad \text{if } \psi^*(y, t) > 0$$

$$\leqslant \lambda(t) \quad \text{if } \psi^*(y, t) = 0 \quad \text{for a.e. } y \in Y, t = 0, 1, \ldots, T. \tag{3.4}$$

Proof of sufficiency. The proof of sufficiency follows that of theorem 8.4.1 in Stone (1975). The essence of this argument is due to D. H. Wagner.

Observe that the concavity of b implies that P_T is a concave functional on F^+. We now proceed to use an argument by contradiction. Suppose that $\psi^* \in \Psi(m)$ satisfies (3.4) and that there is a $\psi \in \Psi(m)$ such that $P_T[\psi] > P_T[\psi^*]$.

Since P_T is concave we have for $0 \leqslant \theta \leqslant 1$,

$$P_T[\psi^* + \theta(\psi - \psi^*)] - P_T[\psi^*] = P_T[(1 - \theta)\psi^* + \theta\psi] - P_T[\psi^*]$$

$$\geqslant (1 - \theta)P_T[\psi^*] + \theta P_T[\psi] - P_T[\psi^*] = \theta(P_T[\psi] - P_T[\psi^*]).$$

It follows that

$$P_T'[\psi^*, \psi - \psi^*] \geqslant P_T[\psi] - P_T[\psi^*] > 0. \tag{3.5}$$

However, by (3.3) and (3.4) we have

$$P'[\psi^*, \psi - \psi^*] = \sum_{t=0}^{T} \int_Y D_T(\psi^*, y, t)[\psi(y, t) - \psi^*(y, t)] \, dy$$

$$= \sum_{t=0}^{T} \int_{\{y \,:\, \psi^*(y, t) > 0\}} D_T(\psi^*, y, t)[\psi(y, t) - \psi^*(y, t)] \, dy$$

$$+ \sum_{t=0}^{T} \int_{\{y \,:\, \psi^*(y, t) = 0\}} D_T(\psi^*, y, t)[\psi(y, t) - \psi^*(y, t)] \, dy$$

$$\leqslant \sum_{t=0}^{T} \int_Y \lambda(t)[\psi(y, t) - \psi^*(y, t)] \, dy = 0, \tag{3.6}$$

where the last equality follows from the fact that

$$\int_Y \psi(y, t) \, dy = \int_Y \psi^*(y, t) \, dy = m(t) \quad \text{for } t = 0, \ldots, T.$$

However (3.6) contradicts (3.5) and sufficiency is proved.

Proof of necessity. Suppose ψ^* is optimal within $\Psi(m)$. Since b is increasing, we observe that ψ^* is also optimal within the larger class obtained by replacing the equality constraint by an inequality constraint in the definition of $\Psi(m)$ in (2.1).

Define

$$L_\gamma[f] = P_T[f] - \sum_{t=0}^{T} \gamma(t)\left[\int_Y f(y, t) \, dy - m(t)\right] \quad \text{for } f \in F^+, \, \gamma \in \mathscr{E}_{T+1}.$$

By theorem 1 of Luenberger (1969, p. 217), there exists $\lambda \in \mathscr{E}_{T+1}^+$ such that $L_\lambda[\psi^*] = \max_{\psi \in F^+} L_\lambda[\psi]$.

For $f \in F^+$ and $h \in K(f)$, the Gateaux differential $L_\lambda'[f, h]$ of L_λ at f in the direction of h exists and is a linear functional of h. In particular by (3.3)

$$L_\lambda'[f, h] = \sum_{t=0}^{T} \int_Y D_T(f, y, t)h(y, t) \, dy - \sum_{t=0}^{T} \lambda(t)\int_Y h(y, t) \, dy. \tag{3.7}$$

Following Luenberger (1969, p. 227), we observe that since $L[\psi^*] = \max_{\psi \in F^+} L[\psi]$, we have for any $\psi \in F^+$

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \{ L_\lambda[\psi^* + \epsilon(\psi - \psi^*)] - L_\lambda[\psi^*] \} \leqslant 0.$$

Thus

$$L_\lambda'[\psi^*, \psi - \psi^*] \leqslant 0 \quad \text{for } \psi \in F^+. \tag{3.8}$$

Setting $\psi = \frac{1}{2}\psi^*$, we obtain

$$0 \geqslant L_\lambda'[\psi^*, -\tfrac{1}{2}\psi^*] = -\tfrac{1}{2}L_\lambda'[\psi^*, \psi^*], \tag{3.9}$$

while setting $\psi = 2\psi^*$ yields

$$L_\lambda'[\psi^*, \psi^*] \leqslant 0. \tag{3.10}$$

(3.7), (3.9), and (3.10) imply

$$\sum_{t=0}^{T} \int_Y [D_T(\psi^*, y, t) - \lambda(t)]\psi^*(y, t) \, dy = 0, \tag{3.11}$$

while (3.8) and (3.11) imply

$$\sum_{t=0}^{T} \int_Y [D_T(\psi^*, y, t) - \lambda(t)] \psi(y, t) \, dy \leqslant 0 \quad \text{for all } \psi \in F^+. \tag{3.12}$$

(3.12) implies that $D_T(\psi^*, y, t) \leqslant \lambda(t)$ for a.e. $y \in Y$, $t = 0, 1, \ldots, T$, and (3.11) implies $D_T(\psi^*, y, t) = \lambda(t)$ for a.e. $y \in Y$ and $t = 0, 1, \ldots, T$ such that $\psi^*(y, t) > 0$. The necessity of the conditions in (3.4) follows, and the theorem is proved.

In the following paragraphs we discuss extensions and specializations of theorem 1.

Discrete space—discrete time. Theorem 1 and the proof given above also hold when the search space Y is discrete provided one interprets $p_t(y)$ in (3.2) as the probability that $X_t = y$.

Continuous time (sufficiency). The obvious analog of the sufficiency part of theorem 1 holds for continuous time provided the target motion process $\{X_t; 0 \leqslant t \leqslant T\}$ has Borel measurable sample paths, the conditional expectation

$$E_{yt}\left[b'\left(\int_0^T \psi(X_s, s) \, ds \right) \right]$$

is well defined, and we take

$$D_T(\psi, y, t) = E_{yt}\left[b'\left(\int_0^T \psi(X_s, s) \, ds \right) \right] p_t(y) \quad \text{for } \psi \in F^+, y \in Y, t \in [0, T]. \tag{3.13}$$

The proof of this assertion parallels the sufficiency proof in theorem 1. The Borel measurability of the sample paths is needed to guarantee that the integral $\int_0^T \psi(X_s, s) \, ds$ is well defined.

Continuous time-continuous space (necessity). The corresponding necessity result for continuous time is more difficult to prove; its proof is not an obvious extension of the one given for theorem 1. However by paralleling the proof of theorem 5.2 in Stone (1977), one may show that the conditions in (3.4) are necessary with D_T defined as in (3.13). In fact that proof shows that for continuous space the necessity result holds when the concavity assumption on b is dropped. The proof in Stone (1977) also allows one to add a constraint of the form $0 \leqslant \psi(y, t) \leqslant B$ for some positive number B with a corresponding change in the conditions in (3.4).

Observe that when $b(z) = 1 - e^{-z}$, we have, for continuous time,

$$D_T(\psi, y, t) = E_{yt}\left[\exp\left(- \int_0^T \psi(X_s, s) \, ds \right) \right] p_t(y). \tag{3.14}$$

If $\{X_s, s \geqslant 0\}$ is a Markov process, then one can show that

$$D_T(\psi, y, t) = \int_X r(x, 0, y, t, \psi) R(y, t, T, \psi) p_0(x) \, dx, \tag{3.15}$$

where under plan ψ, $r(x, 0, y, t, \psi)$ is the probability density that at time t the target is located at y and is undetected given it was at x at time 0, and $R(y, t, T, \psi)$ is the probability that if the target is at point y at time t it will be undetected in the interval $[t, T]$. Because of the Markov nature of the process, $r(x, 0, y, t, \psi) R(y, t, T, \psi)$ is the probability density that a target starting at x at time 0 will pass through the point y at time t and remain undetected throughout $[0, T]$. The integral on the right of (3.15) averages over the distribution of the target's position at time 0 to obtain the probability density that the target passes through point y at time t and remains

undetected throughout $[0, T]$. Now

$$\exp\left(-\int_0^T \psi(X(\omega, s), s)\, ds\right)$$

is the probability of failing to detect the target by time T given it follows path ω. Thus the right-hand side of (3.14) is simply the probability density of the target passing through point y at time t and failing to be detected by time T. From this observation and the above discussion, (3.15) follows.

Thus theorem 5.1 of Saretsalo (1973) and the theorem of Hellman (1972) are special cases of the necessity result obtained in this paper. In fact, the specialization to Markov processes given here is stronger than the result in Saretsalo (1973) in the sense that no assumptions concerning the continuity of the transition function are required. In addition, we have proved sufficiency.

It also follows that the necessary and sufficient conditions obtained in Stone (1977) and Persiheimo (1977) are a special case of the conditions found in this paper. In particular, one can show, in the notation of Stone (1977), that

$$p_t(y) = \int_\Omega \frac{p(\eta_{\omega t}^{-1}(y), \omega)}{J(\eta_{\omega t}^{-1}(y), \omega, t)}\, \gamma(d\omega),$$

and

$$E_{yt}\left[b'\left(\int_0^T \psi(X_s, s)\, ds\right)\right]$$

$$= \frac{1}{p_t(y)} \int_\Omega \frac{p(\eta_{\omega t}^{-1}(y), \omega)}{J(\eta_{\omega t}^{-1}(y), \omega, t)}\, b'\left(\int_0^T \psi(\eta_{\omega s}[\eta_{\omega t}^{-1}(y)], s)\, ds\right)\gamma(d\omega),$$

so that $D_T(\psi, y, t) = E_{yt}[b'(\int_0^T \psi(X_s, s)\, ds)]p_t(y)$ coincides with the definition of D_T in Stone (1977) and the conditions in that reference and in Persiheimo (1977) are a special case of conditions (3.4) provided one makes the obvious changes for the bound B on effort density which is allowed as a constraint in Stone (1977).

Continuous time-discrete space (necessity). The necessity part of theorem 1 also holds for a continuous time and discrete space motion process. Again the proof is not a simple extension of the one given for theorem 1, but one can parallel the proof of theorem 5.2 in Stone (1977) to obtain the result. However, in the case of discrete space, the assumption of concavity for b is required in order to guarantee the necessity of conditions (3.4). The concavity is needed on p. 464 of Stone (1977) where one invokes a Lagrange multiplier result to guarantee the existence of $\lambda(t)$ to satisfy (5.9) at the bottom of that page. In the case of a discrete search space one must invoke a result such as corollary B.1.2 of Stone (1975) which requires the concavity of the detection function b.

Unified statement of results. Most of the above results can be consolidated into a single theorem statement provided we make the appropriate identifications for D_T, p_t, and $[0, T]$. Specifically D_T is given by (3.2) when time is discrete and by (3.13) when time is continuous; p_t is the probability density function for X_t when Y is Euclidean n space, and $p_t(y) = Pr\{X_t = y\}$ when Y is a discrete space. For continuous time F, F^+, $\Psi(m)$, and P_T are defined as in §2 but with integrals replacing summation. In discrete time we understand $[0, T] = \{0, 1, \ldots, T\}$ while in continuous time $[0, T]$ has the usual meaning.

THEOREM 1'. *Suppose b is concave and that it has a bounded nonnegative derivative* *b'. Assume that the sample paths of* $\{X_t, 0 \le t \le T\}$ *are Borel measurable and that* D_T *is well defined for* (y, t) *such that* $p_t(y) > 0$. *Then* ψ^* *is T-optimal within* $\Psi(m)$ *if and only if there exists* $\lambda : [0, T] \to [0, \infty)$ *such that*

$$D_T(\psi^*, y, t) = \lambda(t) \quad \text{if } \psi^*(y, t) > 0$$

$$\le \lambda(t) \quad \text{if } \psi^*(y, t) = 0 \quad \text{for a.e. } (y, t) \in Y \times [0, T]. \quad (3.16)$$

In the case where the search space is $\mathfrak{S}_n$, the necessity of conditions (3.16) remains true when the concavity assumption on b is dropped. In discrete time, the sample paths will always be Borel measurable, and D_T will be well defined for (y, t) such that $p_t(y) > 0$.

4. The special case of an exponential detection function. When the detection function is exponential, we may prove the necessity of the conditions in (3.4) in an elementary manner which requires the use of only the necessary conditions for an optimal stationary target search. The use of the exponential detection functions also allows us to consider the possibility that the detection capability of the search sensor varies over the search space. Specifically we shall assume that for each point $y \in Y$, there is a number $W(y)$ which characterizes the detection performance of the sensor in the neighborhood of y in the sense that if the target is located at y and z effort density is placed there, then $1 - \exp(-W(y)z)$ is the probability of detecting the target. Classically $W(y)$ is called the sweep width of the sensor when operating in the neighborhood of y. For this case

$$P_T[\psi] = 1 - E\left[\exp\left(-\sum_{s=0}^{T} W(X_s)\psi(X_s, s)\right)\right] \quad \text{for } \psi \in F^+, \quad (4.1)$$

and D_T, the kernel of the linear functional $P_T'[\psi, \cdot]$, in (3.3) becomes

$$D_T(\psi, y, t) = E_{yt}\left[\exp\left(-\sum_{s=0}^{T} W(X_s)\psi(X_s, s)\right)\right] p_t(y) W(y)$$

$$\text{for } \psi \in F, y \in Y, t = 0, 1, \ldots, T.$$

THEOREM 2. *Suppose the detection function is exponential and W is bounded. Then* ψ^* *is T-optimal within* $\Psi(m)$, *if and only if there exists* $(\lambda(0), \ldots, \lambda(T)) \in \mathfrak{S}_{T+1}^+$ *such that for* $t = 0, \ldots, T$ *and for a.e.* $y \in Y$,

$$E_{yt}\left[\exp\left(-\sum_{s \ne t} W(X_s)\psi^*(X_s, s)\right)\right] p_t(y) W(y) e^{-W(y)\psi^*(y, t)}$$

$$= \lambda(t) \quad \text{if } \psi^*(y, t) > 0$$

$$\le \lambda(t) \quad \text{if } \psi^*(y, t) = 0. \quad (4.2)$$

PROOF. Since P_T as defined in (4.1) is a concave functional, the sufficiency proof for theorem 1 applies here also.

To prove the necessity part of the theorem, let

$$g_t(y) = E_{yt}\left[\exp\left(-\sum_{s \ne t} W(X_s)\psi^*(X_s, s)\right)\right] p_t(y) \quad \text{for } y \in Y, t = 0, 1, \ldots, T,$$

and

$$G(t) = \int_Y g_t(y)\, dy \quad \text{for } t = 0, 1, \ldots, T.$$

Suppose that for some t (4.2) fails to hold on a set of positive measure in Y. Dividing g_t by $G(t)$ to obtain a probability density $\tilde{g}_t$, we observe that $\psi^*(\cdot, t)$ fails to satisfy the necessary conditions of corollary 2.1.7 of Stone (1975) for $\psi^*(\cdot, t)$ to maximize probability of detection for cost $m(t)$ for a stationary target with probability density $\tilde{g}_t$. Thus $\psi^*(\cdot, t)$ is not optimal for cost $m(t)$ for this stationary target problem.

For nonnegative Borel mea   ole functions f defined on Y, let

$$Q[f] = \int_Y \tilde{g}_t(y)(1 - e^{-W(y)f(y)})\, dy.$$

Then since $\psi^*(\cdot, t)$ is not optimal for cost $m(t)$, we may find an $f^* \geqslant 0$ such that $\int_Y f^*(y)\, dy = m(t)$ and $Q[f^*] > Q[\psi^*(\cdot, t)]$. Observe that

$$1 - P_T[\psi^*] = \int_Y g_t(y)\exp(-W(y)\psi^*(y, t))\, dy$$

$$= 1 - G(t)Q[\psi^*(\cdot, t)] > 1 - G(t)Q[f^*].$$

Thus by taking

$$\psi(y, s) = \begin{cases} \psi^*(y, s) & \text{for } s \neq t, \\ f^*(y) & \text{for } s = t, \end{cases}$$

we have $\psi \in \Psi(m)$ and $P_T[\psi] > P_T[\psi^*]$ which contradicts the assumption that ψ^* is T-optimal within $\Psi(m)$. Thus (4.2) must hold and theorem 2 is proved.

From the definition of $g_t(y)$ and $G(t)$, one can see that $\tilde{g}_t = g_t / G(t)$ is simply the probability density for the target's location at time t given that it was undetected by the search applied at all times other than t. Let $\tilde{\lambda}(t) = \lambda(t) / G(t)$ for $t = 0, 1, \ldots, T$. Then conditions (4.2) become

$$\tilde{g}_t(y)W(y)e^{-W(y)\psi^*(y, t)} = \tilde{\lambda}(t) \quad \text{if } \psi^*(y, t) > 0$$

$$\leqslant \tilde{\lambda}(t) \quad \text{if } \psi^*(y, t) = 0,$$

which are precisely the necessary and sufficient conditions for $\psi^*(\cdot, t)$ to be an optimal allocation of effort for a stationary target with probability density $\tilde{g}_t$. Thus the optimal moving target plan can be characterized in terms of optimal stationary target plans. That is at each time $t = 0, 1, \ldots, T$, the optimal plan ψ^* allocates the effort available at time t so that $\psi^*(\cdot, t)$ maximizes the probability of detecting a stationary target with probability density $\tilde{g}_t$.

This observation was first made by Brown (1978) for the case of discrete time and space target motion and it forms the basis of the numerical calculation of optimal search plans reported in his work. The algorithm has the following form. When no effort has been applied, $\tilde{g}_0 = p_0$, the initial probability distribution for the target's location at time 0. One begins by allocating the effort optimally for p_0 and then computing the posterior distribution $\tilde{g}_1$ for the target's location at time 1 given failure to detect at time 0. Suppose that one has calculated an allocation of effort for $s = 0, 1, \ldots, t - 1$. Then one calculates $\tilde{g}_t$, the probability distribution for the target's location at time t given failure to detect by time $t - 1$, and allocates the effort for time t to be optimal for $\tilde{g}_t$. When one reaches time T, he has computed the myopic search plan, i.e., the plan which at each time step allocates its effort to maximize the increase in detection probability given the effort which has been allocated up to that time.

At this point one begins the second pass by returning to time 0 and calculating $\tilde{g}_0$, the probability density for the target's location at time 0 given that the effort which has been allocated for times $t = 1, \ldots, T$ fails to detect the target. Usually $\tilde{g}_0$ will not be the same as p_0, the initial target location density. The effort allocation for time 0 is readjusted to being optimal for $\tilde{g}_0$. This process is repeated for times $t = 1, \ldots, T$.

This constitutes the second pass. One continues making passes until some stopping criterion is reached. In Brown (1978) it is shown that one can come as close to the optimal plan as he wishes by making a large enough number of passes.

The main obstacle to applying this method of calculating optimal plans is in finding efficient methods of computing $\tilde{g}_t$ for $t = 0, \ldots, T$. This obstacle has been overcome very efficiently by Brown in the case of a discrete time and space Markov process. Stone et al. (1978) presents a numerical optimization method which can be applied to any discrete-time target motion process which can be simulated on a computer.

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